Inverse Problems on the Euclidean Motion Group

Maia Lesosky

Department of Mathematics and Statistics
University of Guelph

11th May 2006

Introduction to Statistical Inverse Problems

- Statistical inverse problems involve 'backwards problems' where we observe data drawn from some probability distribution, for some unknown parameter.
- Most problems that are interesting are 'ill-posed', which is when the inverse operator is unbounded — for example, if the operator is compact.
- Wide ranging area of research - many areas of application, including medical imaging, geophysical applications, automatic image recognition, . . .
- Primary intended application of this research is medical imaging problems, for example, CT scan technology.

Introduction to Statistical Inverse Problems

- Statistical inverse problems involve 'backwards problems' where we observe data drawn from some probability distribution, for some unknown parameter.
- Most problems that are interesting are 'ill-posed', which is when the inverse operator is unbounded — for example, if the operator is compact.
- Wide ranging area of research - many areas of application, including medical imaging, geophysical applications, automatic image recognition,...
- Primary intended application of this research is medical imaging problems, for example, CT scan technology.

Introduction to Statistical Inverse Problems

- Statistical inverse problems involve 'backwards problems' where we observe data drawn from some probability distribution, for some unknown parameter.
- Most problems that are interesting are 'ill-posed', which is when the inverse operator is unbounded — for example, if the operator is compact.
- Wide ranging area of research - many areas of application, including medical imaging, geophysical applications, automatic image recognition,...
- Primary intended application of this research is medical imaging problems, for example, CT scan technology.

Introduction to Statistical Inverse Problems

- Statistical inverse problems involve 'backwards problems' where we observe data drawn from some probability distribution, for some unknown parameter.
- Most problems that are interesting are 'ill-posed', which is when the inverse operator is unbounded — for example, if the operator is compact.
- Wide ranging area of research - many areas of application, including medical imaging, geophysical applications, automatic image recognition,. . .
- Primary intended application of this research is medical imaging problems, for example, CT scan technology.

Background

- $SE(2)$ is the semi-direct product of $\mathbb{R}^2$ and $SO(2)$ (the special orthonormal group on 2 dimensions)
- Can be considered a subgroup of the 3×3 matrices
- Group elements are $g = (R_\theta, \mathbf{r})$ with $R_\theta \in SO(2)$ and $\mathbf{r} \in \mathbb{R}^2$ given by

$$g = (R_\theta, \mathbf{r}) = \begin{pmatrix} \cos\theta & -\sin\theta & r_1 \\ \sin\theta & \cos\theta & r_2 \\ 0 & 0 & 1 \end{pmatrix}$$

- Group operation is matrix multiplication, this group is non-compact and non-commutative

Background

- $SE(2)$ is the semi-direct product of $\mathbb{R}^2$ and $SO(2)$ (the special orthonormal group on 2 dimensions)
- Can be considered a subgroup of the 3×3 matrices
- Group elements are $g = (R_\theta, \mathbf{r})$ with $R_\theta \in SO(2)$ and $\mathbf{r} \in \mathbb{R}^2$ given by

$$g = (R_\theta, \mathbf{r}) = \begin{pmatrix} \cos\theta & -\sin\theta & r_1 \\ \sin\theta & \cos\theta & r_2 \\ 0 & 0 & 1 \end{pmatrix}$$

- Group operation is matrix multiplication, this group is non-compact and non-commutative

Background

- $SE(2)$ is the semi-direct product of $\mathbb{R}^2$ and $SO(2)$ (the special orthonormal group on 2 dimensions)
- Can be considered a subgroup of the 3×3 matrices
- Group elements are $g = (R_\theta, \mathbf{r})$ with $R_\theta \in SO(2)$ and $\mathbf{r} \in \mathbb{R}^2$ given by

$$g = (R_\theta, \mathbf{r}) = \begin{pmatrix} \cos\theta & -\sin\theta & r_1 \\ \sin\theta & \cos\theta & r_2 \\ 0 & 0 & 1 \end{pmatrix}$$

- Group operation is matrix multiplication, this group is non-compact and non-commutative

Background

- $SE(2)$ is the semi-direct product of $\mathbb{R}^2$ and $SO(2)$ (the special orthonormal group on 2 dimensions)
- Can be considered a subgroup of the 3×3 matrices
- Group elements are $g = (R_\theta, \mathbf{r})$ with $R_\theta \in SO(2)$ and $\mathbf{r} \in \mathbb{R}^2$ given by

$$g = (R_\theta, \mathbf{r}) = \begin{pmatrix} \cos\theta & -\sin\theta & r_1 \\ \sin\theta & \cos\theta & r_2 \\ 0 & 0 & 1 \end{pmatrix}$$

- Group operation is matrix multiplication, this group is non-compact and non-commutative

Introduction to Fourier Analysis

- In order to do Fourier analysis we need an operator called an irreducible unitary representation.
- Denote the IUR by $U(g, p)$ where $g \in SE(2)$ and $p \in \mathbb{R}_+$ is an index.
- Represent the operator by an infinite dimensional matrix with matrix elements $u_{mn}(g, p)$.
- Will not distinguish between the operator and the infinite dimensional matrix.

Introduction to Fourier Analysis

- In order to do Fourier analysis we need an operator called an irreducible unitary representation.
- Denote the IUR by $U(g, p)$ where $g \in SE(2)$ and $p \in \mathbb{R}_+$ is an index.
- Represent the operator by an infinite dimensional matrix with matrix elements $u_{mn}(g, p)$.
- Will not distinguish between the operator and the infinite dimensional matrix.

Introduction to Fourier Analysis

- In order to do Fourier analysis we need an operator called an irreducible unitary representation.
- Denote the IUR by $U(g, p)$ where $g \in SE(2)$ and $p \in \mathbb{R}_+$ is an index.
- Represent the operator by an infinite dimensional matrix with matrix elements $u_{mn}(g, p)$.
- Will not distinguish between the operator and the infinite dimensional matrix.

Introduction to Fourier Analysis

- In order to do Fourier analysis we need an operator called an irreducible unitary representation.
- Denote the IUR by $U(g, p)$ where $g \in SE(2)$ and $p \in \mathbb{R}_+$ is an index.
- Represent the operator by an infinite dimensional matrix with matrix elements $u_{mn}(g, p)$.
- Will not distinguish between the operator and the infinite dimensional matrix.

Fourier Transform

- The Fourier transform of a rapidly decreasing function $f \in L^2(SE(2))$ where $g \in SE(2)$, $p \in \mathbb{R}_+$ and its inverse transform are defined as
-
$$\mathcal{F}(f) \equiv \hat{f}(p) = \theta_p = \int_{SE(2)} f(g)U(g^{-1}, p)d(g) \qquad (1)$$
and
-
$$\mathcal{F}^{-1}(\hat{f}) \equiv f(g) = \int_0^{\infty} \operatorname{tr}\left(\hat{f}(p)U(g,p)\right) pdp. \qquad (2)$$

Fourier Transform

- The Fourier transform of a rapidly decreasing function $f \in L^2(SE(2))$ where $g \in SE(2)$, $p \in \mathbb{R}_+$ and its inverse transform are defined as
- $$\mathcal{F}(f) \equiv \hat{f}(p) = \theta_p = \int_{SE(2)} f(g)U(g^{-1}, p)d(g) \qquad (1)$$

  and
- $$\mathcal{F}^{-1}(\hat{f}) \equiv f(g) = \int_0^{\infty} \mathrm{tr}\left(\hat{f}(p)U(g,p)\right) pdp. \qquad (2)$$

Fourier Transform

- The Fourier transform of a rapidly decreasing function $f \in L^2(SE(2))$ where $g \in SE(2)$, $p \in \mathbb{R}_+$ and its inverse transform are defined as
- $$\mathcal{F}(f) \equiv \hat{f}(p) = \theta_p = \int_{SE(2)} f(g) U(g^{-1}, p) d(g) \qquad (1)$$

  and
- $$\mathcal{F}^{-1}(\hat{f}) \equiv f(g) = \int_0^\infty \mathrm{tr}\left(\hat{f}(p) U(g, p)\right) p dp. \qquad (2)$$

- Helpful to look at the matrix elements of the Fourier transform,

$$\hat{f}_{mn}(p) = \langle e^{im\psi}, \hat{f}(p)e^{in\psi} \rangle = \int_{SE(2)} f(g)u_{mn}(g^{-1}, p)d(g), \tag{3}$$

- and the inversion in terms of matrix elements,

$$f(g) = \sum_{n,m\in\mathbb{Z}} \int_0^\infty \hat{f}_{mn}(p)u_{nm}(g,p)pdp. \tag{4}$$

- Recall that the Fourier transform is an infinite dimensional matrix, and the inverse Fourier transform is a single value.

- Helpful to look at the matrix elements of the Fourier transform,

$$\hat{f}_{mn}(p) = \langle e^{im\psi}, \hat{f}(p)e^{in\psi} \rangle = \int_{SE(2)} f(g) u_{mn}(g^{-1}, p) d(g), \tag{3}$$

- and the inversion in terms of matrix elements,

$$f(g) = \sum_{n,m \in \mathbb{Z}} \int_0^\infty \hat{f}_{mn}(p) u_{nm}(g, p) p dp. \tag{4}$$

- Recall that the Fourier transform is an infinite dimensional matrix, and the inverse Fourier transform is a single value.

- Helpful to look at the matrix elements of the Fourier transform,

$$\hat{f}_{mn}(p) = \langle e^{im\psi}, \hat{f}(p)e^{in\psi} \rangle = \int_{SE(2)} f(g)u_{mn}(g^{-1}, p)d(g), \tag{3}$$

- and the inversion in terms of matrix elements,

$$f(g) = \sum_{n,m\in\mathbb{Z}} \int_0^\infty \hat{f}_{mn}(p)u_{nm}(g,p)pdp. \tag{4}$$

- Recall that the Fourier transform is an infinite dimensional matrix, and the inverse Fourier transform is a single value.

- Note that integration over $SE(2)$ can be written as a "double" integral over the subspaces, $SO(2)$ and $\mathbb{R}^2$.

$$\int_{SE(2)} (\cdot) d(g) = \int_{SO(2)} \int_{\mathbb{R}^2} (\cdot) d\mathbf{x} d(\phi). \tag{5}$$

Properties of the Fourier Transform

- The adjoint property:

$$\widehat{f^*}_{mn}(p) = \overline{\widehat{f}_{nm}(p)}$$

where $f^*(g) = \overline{f(g^{-1})}$.

- The convolution property, written symbolically as:

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_2)\mathcal{F}(f_1), \tag{6}$$

and in terms of matrix elements as:

$$\mathcal{F}(f_1 * f_2)_{mn}(p) = \sum_q \hat{f}_{2,mq}(p)\hat{f}_{1,qn}(p)$$

Properties of the Fourier Transform

- The adjoint property:

$$\widehat{f^*}_{mn}(p) = \overline{\widehat{f}_{nm}(p)}$$

where $f^*(g) = \overline{f(g^{-1})}$.

- The convolution property, written symbolically as:

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_2)\mathcal{F}(f_1), \tag{6}$$

and in terms of matrix elements as:

$$\mathcal{F}(f_1 * f_2)_{mn}(p) = \sum_q \hat{f}_{2,mq}(p)\hat{f}_{1,qn}(p)$$

- The $SO(2)$ invariance property: If $f(g) = f(\mathbf{r}) \in L^2(\mathbb{R}^2)$ then

$$\begin{aligned} \hat{f}_{mn}(p) &= \delta_m \tilde{f}_n(-p) = \delta_m \int_{S^1} \tilde{f}(-p\kappa) e^{in\kappa} d(\kappa) \\ &= \delta_m \int_{S^1} \int_{\mathbb{R}^2} f(\mathbf{r}) e^{-i(-p\kappa \cdot \mathbf{r})} d(\mathbf{r}) e^{in\kappa} d(\kappa) \end{aligned}$$

Noise Model

- The noise model is formulated as follows

$$\int_A dY(g) = \int_A \Lambda f(g) d(g) + \varepsilon \int_A dW(g) \tag{7}$$

where $g \in SE(2)$, $f \in \Theta(a, Q) \subset L^2(SE(2))$, $dW(g)$ is Gaussian white noise in $SE(2)$ and $A \subseteq SE(2)$.

- Prefer to work with the 'sequence space model' given by

$$y_p = \Lambda_p \theta_p + \varepsilon \xi_p \tag{8}$$

where $p \in \mathbb{R}_+$.

Noise Model

- The noise model is formulated as follows

$$\int_A dY(g) = \int_A \Lambda f(g)d(g) + \varepsilon \int_A dW(g) \qquad (7)$$

where $g \in SE(2)$, $f \in \Theta(a, Q) \subset L^2(SE(2))$, $dW(g)$ is Gaussian white noise in $SE(2)$ and $A \subseteq SE(2)$.

- Prefer to work with the 'sequence space model' given by

$$y_p = \Lambda_p \theta_p + \varepsilon \xi_p \qquad (8)$$

where $p \in \mathbb{R}_+$.

- The components of the sequence space model are:

$$y_p = \int_{SE(2)} U(g,p)dY(g)$$

and

$$\theta_p = \hat{f}(p) = \int_{SE(2)} \Lambda f(g)U(g^{-1},p)d(g)$$

is the fourier transform over $SE(2)$ of $f(g)$. Note also that

$$\xi_p = \int_{SE(2)} U(g,p)dW(g).$$

— this is 'white noise'.

Minimax Risk Estimation

- The objective is to estimate the unknown θ_p, so we need an estimator and a measure of error.
- Let $H_p y_p$ be a linear estimator of θ_p such that $H = \{H_p : p \in \mathbb{R}_+\}$.
- Define the mean integrated squared risk as

$$R_\varepsilon^\ell(H,\theta) = \int_0^\infty \mathbb{E}||\theta_p - H_p y_p||_p^2 p dp$$
$$= \int_0^\infty \left(\mathrm{tr}\left(\theta_p^t (I - H_p\Lambda_p)^t (I - H_p\Lambda_p)\theta_p\right) + \varepsilon^2 \,\mathrm{tr}(H_p H_p^t)\right) p dp.$$

Minimax Risk Estimation

- The objective is to estimate the unknown θ_p, so we need an estimator and a measure of error.
- Let $H_p y_p$ be a linear estimator of θ_p such that $H = \{H_p : p \in \mathbb{R}_+\}$.
- Define the mean integrated squared risk as

$$\begin{aligned} R_\varepsilon^\ell(H,\theta) &= \int_0^\infty \mathbb{E}||\theta_p - H_p y_p||_p^2 p dp \\ &= \int_0^\infty \left(\operatorname{tr}\left(\theta_p^t (I - H_p \Lambda_p)^t (I - H_p \Lambda_p)\theta_p\right) + \varepsilon^2 \operatorname{tr}(H_p H_p^t)\right) p dp. \end{aligned}$$

Minimax Risk Estimation

- The objective is to estimate the unknown θ_p, so we need an estimator and a measure of error.
- Let $H_p y_p$ be a linear estimator of θ_p such that $H = \{H_p : p \in \mathbb{R}_+\}$.
- Define the mean integrated squared risk as

$$\begin{aligned} R^{\ell}_{\varepsilon}(H,\theta) &= \int_0^{\infty} \mathbb{E}||\theta_p - H_p y_p||_p^2 p dp \\ &= \int_0^{\infty} \left(\mathrm{tr}\left(\theta_p^t (I - H_p\Lambda_p)^t (I - H_p\Lambda_p)\theta_p\right) + \varepsilon^2 \,\mathrm{tr}(H_p H_p^t)\right) p dp. \end{aligned}$$

- Define the linear minimax risk as

$$r_\varepsilon^\ell(\Theta) = \inf_H \sup_{\theta\in\Theta} R_\varepsilon^\ell(H,\theta). \tag{9}$$

- One goal of this research is to calculate the exact (i.e. including the constant) linear minimax risk.

- Define the linear minimax risk as

$$r_{\varepsilon}^{\ell}(\Theta) = \inf_{H} \sup_{\theta \in \Theta} R_{\varepsilon}^{\ell}(H, \theta). \tag{9}$$

- One goal of this research is to calculate the exact (i.e. including the constant) linear minimax risk.

Introduction to Radon Transform

- The Radon transform forms the backbone of most medical imaging techniques.
- Transform two dimensional images with lines into a domain of possible line parameters.
- Want to use the Fourier transform on $SE(2)$ and the projection slice theorem to represent the Radon transform as a convolution integral.
- The Radon transform is given by

$$\mathcal{R}f(r,\theta) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, x_2)\delta(r - x_1 \cos\theta - x_2 \sin\theta) dx_1 dx_2 \quad (10)$$

Introduction to Radon Transform

- The Radon transform forms the backbone of most medical imaging techniques.
- Transform two dimensional images with lines into a domain of possible line parameters.
- Want to use the Fourier transform on $SE(2)$ and the projection slice theorem to represent the Radon transform as a convolution integral.
- The Radon transform is given by

$$\mathcal{R}f(r,\theta) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, x_2) \delta(r - x_1 \cos\theta - x_2 \sin\theta) dx_1 dx_2 \quad (10)$$

Introduction to Radon Transform

- The Radon transform forms the backbone of most medical imaging techniques.
- Transform two dimensional images with lines into a domain of possible line parameters.
- Want to use the Fourier transform on $SE(2)$ and the projection slice theorem to represent the Radon transform as a convolution integral.
- The Radon transform is given by

$$\mathcal{R}f(r,\theta) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, x_2)\delta(r - x_1 \cos\theta - x_2 \sin\theta) dx_1 dx_2 \quad (10)$$

Introduction to Radon Transform

- The Radon transform forms the backbone of most medical imaging techniques.
- Transform two dimensional images with lines into a domain of possible line parameters.
- Want to use the Fourier transform on $SE(2)$ and the projection slice theorem to represent the Radon transform as a convolution integral.
- The Radon transform is given by

$$\mathcal{R}f(r,\theta) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x_1,x_2)\delta(r - x_1\cos\theta - x_2\sin\theta)dx_1dx_2 \quad (10)$$

Radon/Convolution

- We can write the Radon transform of a real valued function f as a convolution integral over $SE(2)$ as follows:

$$\mathcal{R}(f) \equiv \mathcal{R}f(\kappa, a_1) = (\Delta * f^*)(g) = \int_{SE(2)} \Delta(gh)f(h)d(h) \tag{11}$$

where $f^*(h) = \overline{f(h^{-1})}$, $\Delta(h) = \delta(\mathbf{b} \cdot \mathbf{e_1})$, $f(h) = f(\mathbf{b})$, and $\kappa = -A^{-1}e_1$, $a_1 = \mathbf{a} \cdot \mathbf{e_1}$.

Discussion

- Have a solution to the exact minimax risk calculation — still need to prove convergence.
- The next step is to begin programming to do a simulation study — then application to real data sets.

Discussion

- Have a solution to the exact minimax risk calculation — still need to prove convergence.
- The next step is to begin programming to do a simulation study — then application to real data sets.